# SIMULATION OF OPTIMAL SPEED CONTROL FOR A DC MOTOR USING LINEAR QUADRATIC REGULATOR (LQR)

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## ABSTRACT:

This paper describes DC motor speed control based on optimal Linear Quadratic Regulator (LQR) technique. Controller's objective is to maintain the speed of rotation of the motor shaft with a particular step response.

The controller is modeled in MATLAB environment, the simulation results show that the proposed controller gives better performance and less settling time when compared with the traditional PID controller.

## السيطرة المثالية على سرعة محرك التيار المستمر باستخدام مقوم خطي من الدرجة الثانية

## الخلاصة:

لقد تم تقديم مسيطر باستخدام تقنية المقوم الخطي ذو الدرجة الثانية LQR واستخدم هذا المسيطر للسيطرة على سرعة محرك التيار المستمر علما ان الهدف المرجو من استخدام المسيطرات هو الحفاظ على سرع مستقرة وثابتة للمحرك.

لقد تم التحليل باستخدام برنامج MATLAB حيث اثبتت النتائج ان هذا النوع من المسيطرات يعطي اداء احسن واستقرار الوقت فيه يكون اقل مقارنة مع المسيطر PID التقليدي.

**Keywords: DC motor, Optimal Speed control, PID controllers, LQR**

## INTRODUCTION:

Due to the excellent speed control characteristics of a DC motor, it has been widely used in industry (such as cars, trucks and aircraft) even though its maintenance costs are higher than the induction motor. As a result, authors have paid attention to position control of DC motor and prepared several methods to control speed of such motors. Proportional–Integral-Derivative (PID) controllers have been widely used for speed and position control. [Neenu, 2009]

They designed a position controller of a DC motor by selection of PID parameters using genetic algorithm (GA) once and secondly by using Ziegler and Nichols method of tuning the parameters of PID controller. They found that the first method gives better results than the second one.

[Delavari Hadi, 2006], presented and compared two types of controllers which are PID controller and optimal controller. The PID compensator is designed using (GA), while the other compensator is made optimal and integral state feedback controller with Kalman filter. Computer simulations have been carried out. Finally they found that the second controller gives less settling, less overshoot and better performance encountering with noise and disturbance parameters variations.

Other authors like [Boumediene, 2009], used a particle swarm optimization (PSO) instead of (GA). They presented a PID controller based on (PSO) method of tuning controller parameters. They modeled their PID-PSO controller in MATLAB environment and compare the results with fuzzy logic controller (FLC) using PSO. They found that PID-PSO controller gives better performance and minimal rise time than FLC-PSO controller.

[Sharaf, 2007], presented a novel PID dual loop controller for a solar photovoltaic (PV) powered industrial permanent magnet DC (PMDC) motor drive. MATLAB/SIMULINK was used in the analysis for the GUI environment.

[Molavi and khaburi, 2008], introduced the optimal strategies for speed control of permanent magnet synchronous motor (PMSM) through the linear quadratic regulator (LQR) and linear quadratic Gaussian (LQG) methodologies.
The simulation results showed that the proposed controllers have better performance for the sake of design criteria like overshoot and settling time of the step response.

[Gwo, 2004], presented a novel optimal PID controller using (LQR) methodology in tuning the parameters of PID controller. The new PID controller is applied to control the speed of brushless DC motor (BLDC). Finally, the computer simulation and experimental results showed that the proposed controller gives better performance than the traditional controller.

This paper presents LQR controller which applied to control the speed of a DC motor. The rest of the paper is presented as follows: at first the plant model is described. The next section describes the PID technique and the design of LQR. Then simulation results are presented. Finally, the last section contains paper conclusion.

## PLANT MODEL:

The speed of a DC motor is proportional to the voltage applied to it. While, its torque is proportional to the motor current. Speed control can be achieved by variable battery tappings, variable supply voltage, resistors or electronic controls.
A simple motor model is shown in Fig.1. The armature circuit consist of a resistance (Ra) connected in series with an inductance (La), and a voltage source ($e_b$) representing the back emf (back electromotive force) induced in the armature when during rotation. [Ogata, 1998 and 2002].

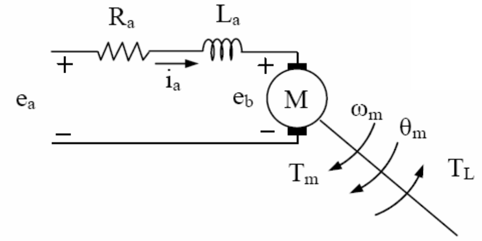

**Fig.1 DC-Motor Model**

The motor torque $T_m$ is related to the armature current, $i_a$ , by a torque constant $K_i$;

$$T_m = K_i i_a \tag{1}$$

The back emf, $e_b$, is relative to angular velocity by;

$$e_b = k_b \omega_m = k_b \frac{d\theta}{dt} \qquad (2)$$

From Fig. 1 we can write the following equations based on the Newton's law combined with the Kirchoff's law:

$$L_a \frac{di_a}{dt} + R_a i_a = e_a - K_b \frac{d\theta}{dt} \qquad (3)$$

$$J_m \frac{d^2\theta}{dt^2} + B_m \frac{d\theta}{dt} = K_i i_a \qquad (4)$$

There are several different ways to describe a system of linear differential equations. The plant model will be introduced in the form of **state-space representation** and given by the equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (5)$$

According to eq.s from (2) to (4), the state space model will be:

$$\begin{bmatrix} \dot{i}_a \\ \dot{\omega}_m \\ \dot{\theta}_m \end{bmatrix} = \begin{bmatrix} -R_a/L_a & -K_b/L_a & 0 \\ K_i/J_m & -B_m/J_m & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} e_a \qquad (6)$$

$$\omega_m = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix} \qquad (7)$$

with the block diagram as shown in Fig. 2

The DC motor data taken for this work are [Nguyen, 2006]:

| Symbol | Value and unit |
|---|---|
| E | = 12volt |
| $J_m$ | = 0.01kgm$^2$ |
| $B_m$ | = 0.00003kgm$^2$/s |
| $K_i$ | = 0.023Nm/A |
| $K_b$ | = 0.023V/rad/s |
| $R_a$ | = 1Ω |
| $L_a$ | = 0.5H |

## DESIGN OF THE LQR CONTROLLER [Firas, 2006]

LQR control that designed is classified as optimal control systems. This is an important function of control engineering. Fig.3 shows the designed LQR state-feedback configuration.

The purpose of the design is to realize a system with practical components that will provide the desired operating performance. The desired performance can be readily stated in terms of time domain performance indices. For example, the maximum overshoot and rise time for a step input are valuable time domain indices. In the case of steady state and transient performance, the performance indices are normally specified in the time domain.

The performance of a control system can be represented by integral performance measures. Therefore, the design of the system must be based on minimizing a performance index, such as the integral of the squared error (ISE).

The specific form of the performance index can be given as in eq.(8), where $x^T$ indicates the transpose of the $x$ matrix, then, in terms of the state vector, is

$$J = \int_0^{t_f} (x^T x)dt \qquad (8)$$

Where $x$ equals the state vector, and $t_f$ equals the final time.

Then the design steps are as follows:

1- *Determine the matrix P that satisfies eq.(8-a), where H is known.*

$$\boxed{H^T P + PH = -I} \qquad (8\text{-a})$$

2- *Minimize **J** by determining the minimum of eq.(8-b) by adjusting one or more unspecified system parameters* (Firas, 2006 )

$$\boxed{J = \int_0^{\infty} x^T x dt = x^T(0) P x(0)} \qquad (8\text{-b})$$

Upon examining the performance index (eq.8), recognizing that the reason the magnitude of the control signal is not accounted for in the original calculation is that ***u*** (equals the control vector) is not included within the expression for the performance index. However, in many cases, the expenditure of the control signal energy are concerned. For example, in an electric vehicle control system, ***(u)***$^2$ represents the expenditure of battery energy and must be restricted to conserve the energy for long periods of travel. To account for the expenditure of the energy of the control signal, it will be utilize the performance index

$$J = \int_0^\infty (x^T I x + \lambda u^T u) dt \qquad (9)$$

Where $\lambda$ is the scalar weighting factor and $I$ = identity matrix. The weighting factor $\lambda$ will be chosen so that the relative importance of the state variable performance is contrasted with the importance of the expenditure of the system energy resource that is represented by $u^T u$ as in the previous paragraphs, the state variable feedback will be represented by the matrix equation

$$u = -Kx \qquad (10)$$

And the system with this state variable feedback as

$$\dot{x} = Ax + Bu = Hx \qquad (11)$$

Now, substituting eq.(10) into eq.(9), then

$$J = \int_0^\infty (x^T I x + \lambda (Kx)^T (Kx) dt$$

$$= \int_0^\infty [x^T (I + \lambda K^T K) x] dt = \int_0^\infty x^T Q x dt \quad (12)$$

Where Q= $(I + \lambda K^T K)$ is an $n \times n$ matrix.

postulating the existence of an exact differential so that

$$\frac{d}{dt}(x^T P x) = -x^T Q x \qquad (13)$$

Then, in this case, it is required that

$$H^T P + PH = -Q \qquad (14)$$

As before in eq.(8-a)

$$J = x^T (0) P x(0) \qquad (15)$$

Now, the design steps are exactly as for eq.(8-a) and eq.(8-b) with exception that the left side of eq.(14) equals –Q instead of –I. Of course, if $\lambda = 0$, eq.(14) reduces to eq.(8-b).

Consider the single-input, single-output (SISO) system with

$$\dot{x} = Ax + Bu \qquad (11)$$

and feedback

$$u = -Kx = -[k_1 \quad k_2 \ldots k_n] x$$

The performance index is

$$J = \int_0^\infty (x^T Q x + R u^2) dt$$

or the performance index is

$$J = \int_0^\infty [(tracking \quad errors)^2 + (weighted \quad inputs)^2] dt$$

where $R$ is the scalar weighting factor. This index is minimized when

$$K = R^{-1} B^T P$$

The $n \times n$ matrix p is determined from the solution of equation

$$A^T P + PA - PBR^{-1} B^T P + Q = 0 \qquad (16)$$

The two matrices ***Q*** and ***R*** are selected by design engineer by tray and error. Generally speaking, selecting ***Q*** large means that, to keep ***J*** small. On the other hand selecting ***R*** large means that the control input ***u*** must be smaller to keep ***J*** small. One should select ***Q*** to be *positive semi-definite* and ***R*** to be *positive definite*. This means that the scalar quantity $x^T Q x$ is always positive or zero at each time ***t***. And the scalar quantity $Ru^2$ is always positive at each time ***t*** .

eq.(16) can be easily programmed for a computer, or solved using MATLAB. eq.(16) is often called the **Riccati Equation**. This optimal control called the **Linear Quadratic Regulator (LQR)** which is shown in state-space configuration in Fig. 3. Combine Fig.2 with Fig.3 yields Fig.4, which shows the use of LQR controller with the DC motor.

## SIMULATION MATLAB SCRIPT FILE AND RESULTS:

The simulation procedure may be summarized as follows:

- First input the DC motor data,
- Write the differential equations for the model then get the state space representation as in eq. (6)
- Get the open loop transfer function and the closed loop step response
- Finally performing the performance of PID controller and LQR controller and compare the results.
- The output will be taken as $\omega_m(s)$ from Fig.4.

**MATLAB script file is:**

```
% file name: DCM2.m
clear
clc
t = 0:0.001:10;
% DATA1============
J = 0.01;
b = 0.00003;
K = 0.023;
R = 1;
L = 0.5;
A = [-b/J   K/J
     -K/L   -R/L];

B = [0
     1/L];
C = [1   0];
D = 0;
% sys = ss(A,B,C,D);
num=K;
den=[(J*L) ((J*R)+(L*b)) ((b*R)+K^2)];
open=tf(num,den);
closed= feedback(open,1)
%============PID===============
Kp = 150;  %
Ki = 150;  % took by try
Kd = 0.4;  %
PID = tf([Kd Kp Ki],[1 0]);
PIDsys = feedback(PID*open,1);
% +++++++++++++++++++++++++++++++++++%
```

```
  % Linear Quadratic Regulator design
                LQR
% %+++++++++++++++++++++++++++++++++++%
Q=[.2 0;0 0.028];
R=[.2];
[KK,S,e] =lqr(A,B,Q,R)
ZZ=(A-B*KK);
LQR=ss(ZZ,B,C,D);
damp(LQR)
[num1,den1]=ss2tf(ZZ,B,C,D,1);
G=tf(num1,den1)    %ALWAYS den=1
BECAUSE ONLY u
%++++++++++++++++++++++++++++++++++++%
% Linear Quadratic Regulator design
         LQR Step
%++++++++++++++++++++++++++++++++++++%
figure(1)
step(closed,t),title('Closed Loop step
response')
xlabel('Time','FontSize',11);
ylabel('P.U. speed','FontSize',11);
figure(2)
step(PIDsys,t),title('PID step
response')
xlabel('Time','FontSize',11);
ylabel('P.U. speed','FontSize',11);
figure(3)
step(LQR,t),title('LQR step response')
xlabel('Time','FontSize',11);
ylabel('P.U. speed','FontSize',11);
figure(4)
step(PIDsys,LQR,closed,t),title('step
all')
xlabel('Time','FontSize',11);
ylabel('P.U. speed','FontSize',11);
```

**Finding the poles and zeros for a transfer function in MATLAB :**

To find the zeros use the command: `z=zero(g)`
To find the poles use the command: `p=pole(g)`
where g: is the transfer function.

After executing the previous script file the open loop transfer function of the DC motor according to the selected data is:

```
ωm(s)               0.023
------- = -------------------------
ea(s)    0.005s^2+0.01002s+0.000559
```

The two poles for the open loop transfer function are: `(-0.0575+j0)` and `(-1.95+j0)`.

The closed loop for speed of the DC motor with unity feedback transfer function is:

$$\frac{\omega_m(s)}{e_a(s)} = \frac{0.023}{0.005s^2+0.01002s+0.02356}$$

with two poles: $(-1+j1.92)$ and $(-1-j1.92)$.

The closed loop step response for speed is shown in Fig.5, with settling time of 3.83sec. and peak amplitude of 1.17.

Now by using PID controller for a DC motor we get better settling time (2.76 sec) and the peak amplitude (1.84) than closed loop response as shown in Fig.6

Using LQR controller for a DC motor gives better settling time (1.99 sec) and peak amplitude (1) than PID controller response as shown in Fig.7 The LQR transfer function is:

$$\frac{\omega_m(s)}{e_a(s)} = \frac{4.6}{s^2 + 3.681s + 4.601}$$

The closed loop, PID and LQR step responses are plotted on the same figure as shown in Fig.8.

## CONCLUSION

Speed control of a DC motor is an important issue, so this paper presents a design method to determine the optimal speed control using LQR method. The obtained results showed that the presented controller has shorter settling time and smaller overshoot than that of the traditional PID controller as shown in Table 1.

**Table (1) Simulation Results**

| | Settling Time | Peak Amplitude | Over shoot |
|---|---|---|---|
| Closed Loop With Unity Feedback | 3.83 sec | 1.17 | 19.5% |
| PID Controller | 2.76 sec | 1.84 | 84.8% |
| LQR Controller | 1.99 sec | 1 | 0.525% |

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## LIST OF SYMBOLS:

$A =$ n×n constant matrix
$B =$ n×1 constant matrix
$B_m =$ viscous friction coefficient (kgm$^2$/s)
$C =$ 1×n constant matrix
$D =$ constant
$e_a(t)$ = applied voltage (V)
$e_b(t)$ = back emf (V)
$i_a(t)$ = armature current (A)
$J_m$ = moment of inertia of rotor (kg.m$^2$)
$K_b$ = back emf constant (V/rad/s) (Kb =Ki )
$K_i$ = torque constant (Nm/A)
$L_a$ = armature inductance (H)
$R_a$ = armature resistance (Ω)
$t_f$= final time(sec)
$T_L(t)$ = load torque (Nm)
$T_m(t)$ = motor torque (Nm)
$u =$ control signal
$x =$ state vector
$y =$ output signal
$\theta_m$ (t) = rotor displacement (rad)
$\omega_m$ (t)= rotor angular velocity (rad/s)

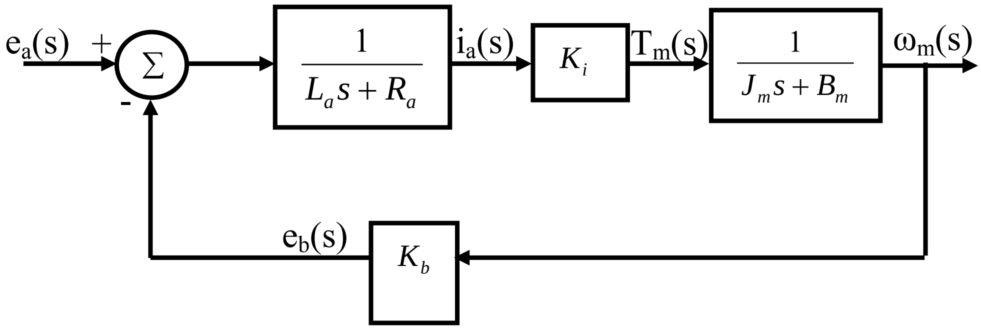

**Fig. 2- DC-Motor System Block Diagram for speed (Plant System)**

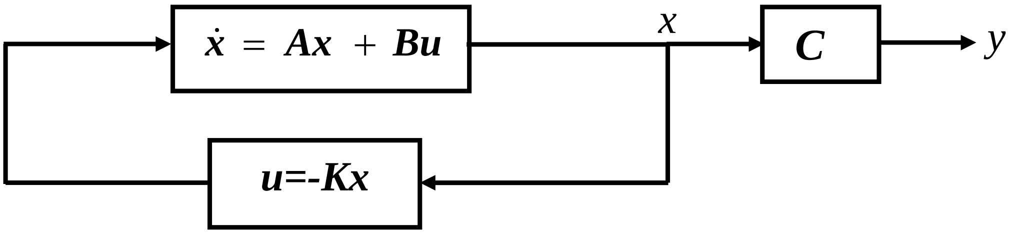

**Fig. 3 Linear Quadratic Regulator Structure**

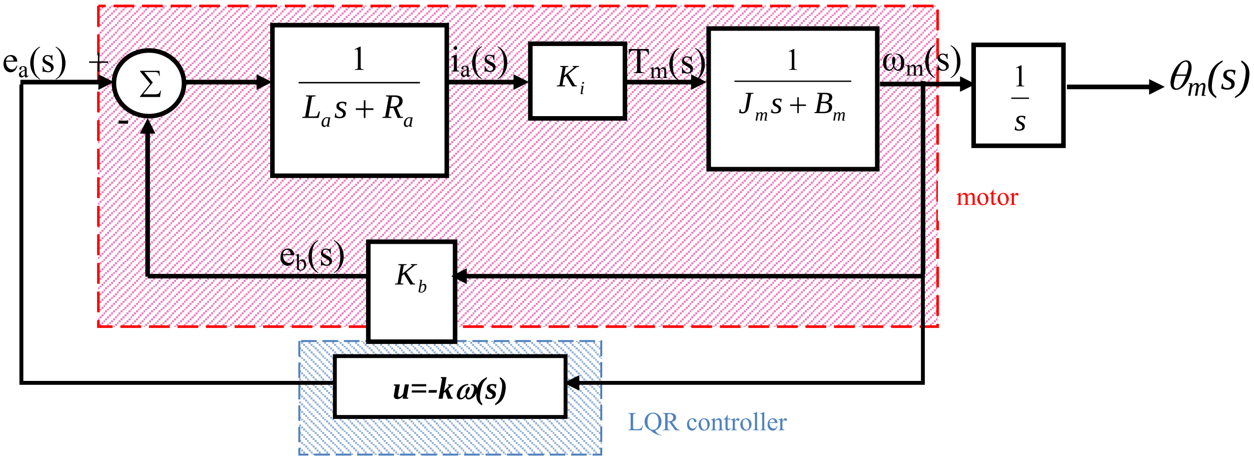

**Fig. 4- DC-Motor System with LQR**

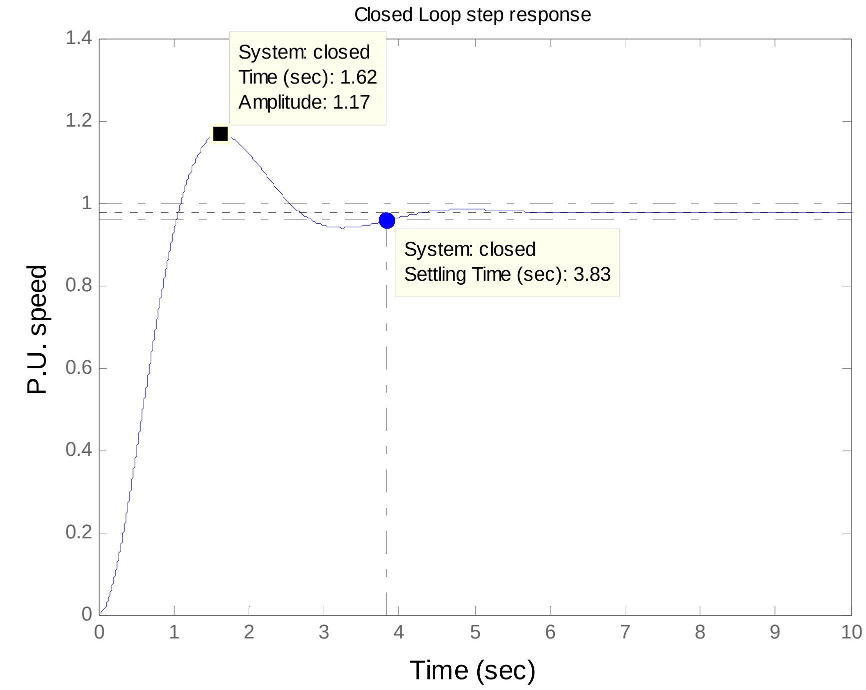

**Fig. 5 Closed Loop Step Response**

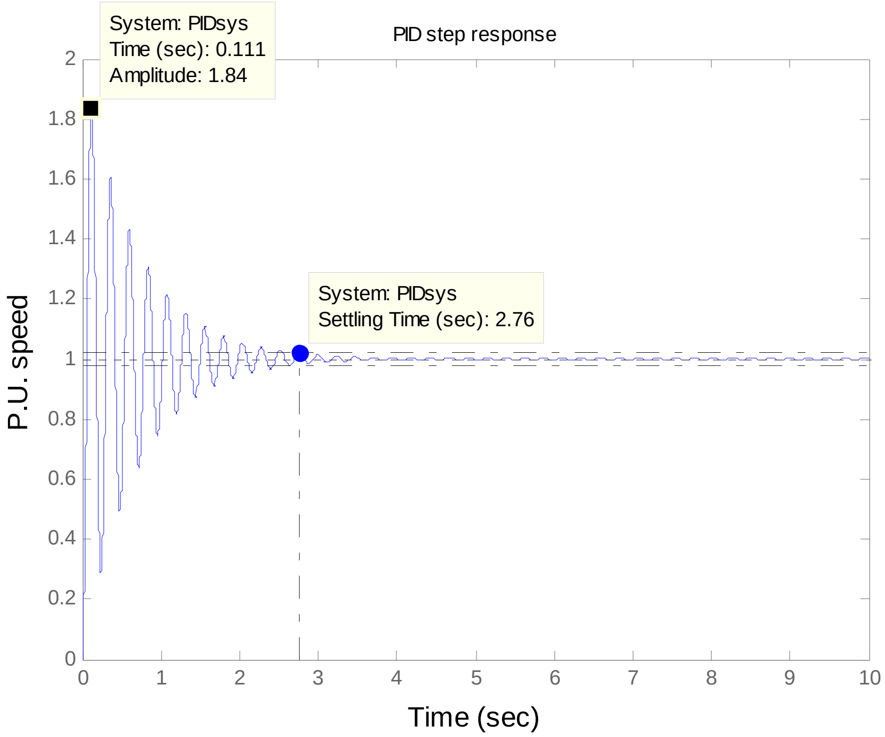

**Fig.6 PID Step Response**

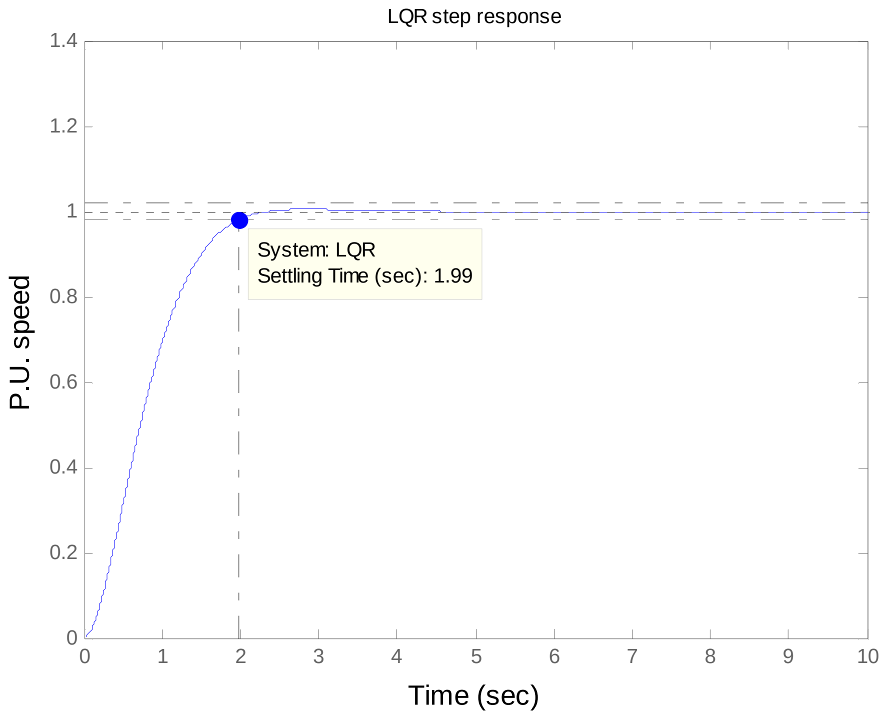

**Fig.7 LQR Step Response**

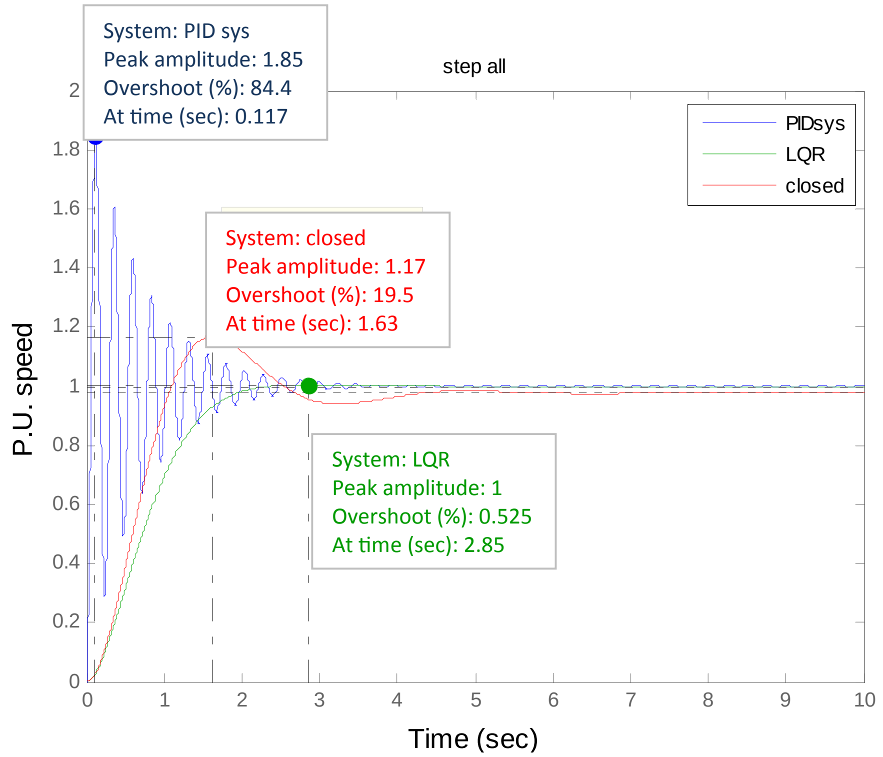

**Fig.8 Closed loop, PID and LQR Step response**