# ELASTIC STABILITY OF FRAMES HAVING CONCAVE TAPERED STRUTS

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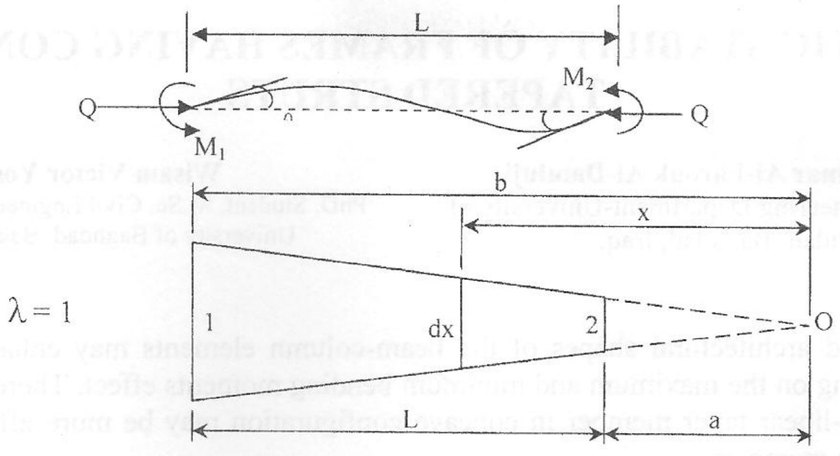
## ABSTRACT
The aesthetic and architectural shapes of the beam-column elements may enhance the elements strength depending on the maximum and minimum bending moments effect. Therefore, the stability value of the non-linear taper member in concave configuration may be more efficient than linear taper or prismatic members.
The modified stability functions will be obtained from the solution of the basic differential equation, where this basic differential equation depends on the non-linearity factor [1] $\lambda$ of the beam-column shape and the shape factor of the cross-sectional area. These two factors, multiplied by others, produce the modified shape factor, which affects the results of the basic differential equation solutions.

## الخلاصة
إنّ الأشكال الفنية والمعمارية للعناصر المحملة بأحمال محورية ممكن أن تزيد من قوة هذه العناصر عند توافقها مع العزوم القصوى والدنيا. لذلك فان قيمة الأستقرارية للعناصر الموشورية لاخطياً كما في الهيئة المقعرة ربما تكون اكثر كفاءة من العناصر الموشورية خطياً أو المنتظمة المقطع.
إنّ الدالة المعدلة للأستقرارية سوف يتم إيجادها من حل المعادلة التفاضلية الأساسية المستندة إلى معامل اللاخطية $\lambda$ للأشكال المحملة أحمال محورية ومعامل المقطع. إنّ حاصل ضرب هذين المعاملين ينتج معامل المقطع المعدل والذي يؤثر على نتائج الحلول للمعادلة التفاضلية الأساسية.

## KEY WORDS
Beam-column, Stability function, Non-linear problems, Non-prismatic members, Shape factor, Concave sections.

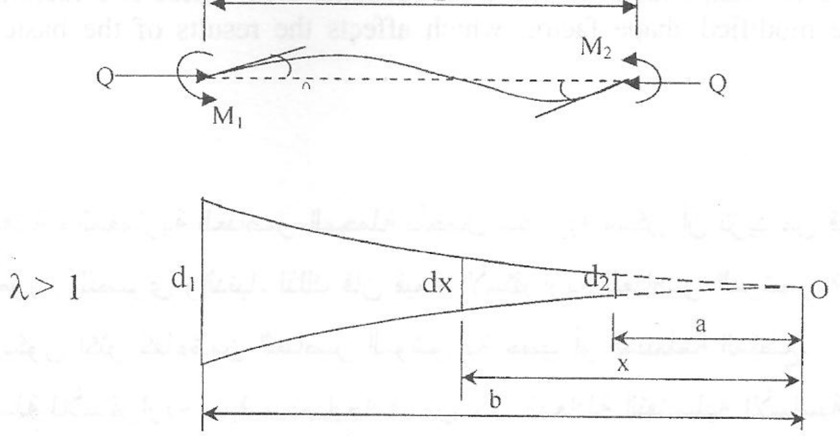
## INTRODUCTION
Present day demands for economy of materials together with sufficient strength leads to the use of more slender structural members and thus a greater understanding of the stability behavior becomes essential. Increasing the second moment of area for each strut, which can very often be accomplished by increasing the cross-sectional dimensions, can enhance the strength of the structure.
The increase of depth, width or both dimensions of struts is followed by and associated to the increase of the subjected bending moment and axial force. The configuration of the struts may be

increasing with cross-sectional dimension along the axis of the member producing a linearly or non-linearly tapered member **Fig. (1)**.




a. Linear non-prismatic

Fig. (1) Tapered beam-column element




b. Non-linear (concave)

Fig. (1) Tapered beam-column element (continued)

## MODIFIED STABILITY FUNCTIONS AT COMPRESSIVE AND TENSILE AXIAL LOAD

The modified slope-deflection equations associated with an elastic prismatic strut for a non-sway mode are:

$$M_1 = \frac{EI}{L}(S\theta_1 + SC\theta_2) \quad (1)$$

$$M_2 = \frac{EI}{L}(SC\theta_1 + S\theta_2) \quad (2)$$

The stability functions S and SC are functions of the non-dimensional parameter $\rho = Q/Q_E$, where Q is the constant axial load and $Q_E$ is the Euler load.

The modified stability functions of the non-linear, non-prismatic members in a concave tapered configuration are given by Equations (3) and (4) they are different at the ends.

$$M_1 = \frac{EI_2}{L}\left(S_1\theta_1 + \overline{SC}\,\theta_2\right) \tag{3}$$

$$M_2 = \frac{EI_2}{L}\left(\overline{SC}\,\theta_1 + S_2\,\theta_2\right) \tag{4}$$

where $I_2$ is the moment of inertia at lower depth of member, $S_1$, SC and $S_2$ are the modified stability functions of the non-dimensional parameter $\rho_2 = Q/Q_E$.
The derivation of the modified stability functions by the exact method is presented here for non-linear tapered members in either one or two directions.
The depth d (x) may be expressed by:

$$d(x) = d_2 (x/a)^{\lambda} \tag{5}$$

where a is the distance of end 2 from the origin O (point of zero), and $d_2$ is the depth at end 2 **Fig. (1)**.

From Equation (5), the depth at end 1 can be obtained as: –

$$d_1 = d_2 (b/a)^{\lambda} \tag{6}$$

where b is the distance of end 1 from the origin O. Writing $U^{\frac{1}{\lambda}} = a/b$ or

$$\overline{U} = a/b \tag{7}$$

where U is the modified taper ratio and may be obtained as: –

$$\overline{U} = U^{\frac{1}{\lambda}} \tag{8}$$

The moment of inertia of the strut may be expressed in the form shown by Equation (9):

$$I(x) = I_2 (x/a)^{\lambda m} \tag{9}$$

where I(x) is the moment of inertia of the section at distance x from the origin O. Equation (9) can be written as:

$$I(x) = I_2 (x/a)^{\overline{m}}$$

where $\overline{m}$ is the modified shape factor and may be obtained as in Equation (10) ( Al-Damarchi, 1999)

$$\overline{m} = \lambda m \tag{10}$$

where m is the shape factor for linear taper members and may be obtained by using Equation (11):

$$m = \log(I_2 / I_1)/\log U \tag{11}$$

The basic differential equation of the beam subjected to a constant axial force Q is ( Mclachlan.1961): –

$$EI(x)\frac{d^2y}{dx^2} \pm Qy = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \tag{12}$$

Substituting Equation (9) into Equation (12) yields:

$$EI_2\left(\frac{x}{a}\right)^{\overline{m}} \frac{d^2y}{dx^2} \pm Qy = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \tag{13}$$

The right hand side of Equation (13) can be reduced to zero by replacing "y" by "Z" when

$$Z = y - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b). \tag{14}$$

Thus, the differential equation becomes:

$$\frac{d^2Z}{dx^2} \pm \omega^2 x^{-\overline{m}} Z = 0 \tag{15}$$

where

$$\omega^2 = \pm Qa^{\overline{m}}/EI_2\,. \tag{16}$$

Equation (15) can be transformed into a Bessel Equation by the form shown in Equation (17) (Al-Sarraf, 1979)

$$\frac{d^2Z}{dx^2} - \frac{(2\alpha-1)}{x}\cdot\frac{dZ}{dx} + \left(\beta^2\gamma^2x^{2\gamma-2} + \frac{\alpha^2 - n^2\gamma^2}{x^2}\right)Z = 0\,. \tag{17}$$

This equation has a general solution of ( Mclachlan, 1961)

$$Z = x^{\alpha}\left[AJ_n\left(\beta x^{\gamma}\right) + BJ_{-n}\left(\beta x^{\gamma}\right)\right] \tag{18-a}$$

or

$$Z = x^{\alpha}\left[AI_n\left(\beta x^{\gamma}\right) + BI_{-n}\left(\beta x^{\gamma}\right)\right] \tag{18-b}$$

where A and B are constants of integration, if epending on (n) if not an integer; then $J_n$ and $I_n$ are Bessel and the modified Bessel functions of the first kind of order (n). Therefore, the solution can be written down in terms of Bessel functions by giving particular values to the constants $\alpha, \beta, \gamma$ and n. By comparing the two Equations (15) and (18), the constants $\alpha, \beta, \gamma$ and n can be obtained:

$$\alpha = 0.5 \ , \ \beta = \pm\frac{2\omega}{2-\overline{m}} \ , \ \gamma = \frac{2-\overline{m}}{2} \ , \ n = \pm\frac{1}{2-\overline{m}}$$

Hence, the general solution of the fundamental Equation (12) is:

$$y = \sqrt{x}\left[AJ_n\left(\beta x^{\gamma}\right) + BJ_{-n}\left(\beta x^{\gamma}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \tag{19-a}$$

or

$$y = \sqrt{x}\left[AI_n\left(\beta x^{\gamma}\right) + BI_{-n}\left(\beta x^{\gamma}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \tag{19-b}$$

depending on whether Q is compression or tension.
There are four unknowns A, B, $M_1$, and $M_2$, which have to be determined from the following boundary conditions:

at $x = a$, $y = 0$ and $dy/dx = \theta_2$,
and $x = b$, $y = 0$ and $dy/dx = \theta_1$.

The solution of the basic differential equation of an elastic curve for nonlinear non-prismatic members depends on the value of the shape factor m and non-linearity factor $\lambda$.
In fact, the non-linear non-prismatic members can be classified in three types according to the non-linearity factor $\lambda$:

1- $\lambda > 1$, the configuration of the beam element is a concave tapered member along its axis.
2- $\lambda = 1$, the configuration of the beam element is a linear tapered member along its axis.
3- $\lambda < 1$, the configuration of the beam element is a convex tapered member along its axis.

The non-linearity factor $\lambda > 1.0$ in a concave taper member is considered here.

**Concave Taper for m = 4 and $\lambda$=2.6**

$$\overline{m} = m\overline{\lambda} = 10.4$$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5 \ , \ \gamma = \frac{2-10.4}{2} = -4.2 \ , \ n = \frac{1}{2-10.4} = \pm 0.119 \ , \ \beta = \frac{2\omega}{2-10.4} = \pm 0.238\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AJ_{0.119}\left(\frac{0.238\omega}{x^{4.2}}\right) + BJ_{-0.119}\left(\frac{0.238\omega}{x^{4.2}}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \tag{20}$$

$$\frac{dy}{dx}=\frac{\omega}{x^{4.7}}AJ_{1.119}\left(\frac{0.238\omega}{x^{4.2}}\right)-\frac{\omega}{x^{4.7}}BJ_{-1.119}\left(\frac{0.238\omega}{x^{4.2}}\right)+\frac{M_1+M_2}{QL} \tag{21}$$

The values of the constants A and B are obtained by substituting the boundary conditions in Equation (20):

$$A=\frac{M_1J_{-0.119}(\alpha)\sqrt{a}+M_2J_{-0.119}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \tag{22}$$

$$B=-\frac{M_1J_{0.119}(\alpha)\sqrt{a}+M_2J_{0.119}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \tag{23}$$

where

$$Z=J_{0.119}(\alpha)J_{-0.119}(\beta)-J_{-0.119}(\alpha)J_{0.119}(\beta) \tag{24}$$

$$\alpha=0.238\frac{\omega}{a^{4.2}}\quad,\quad\beta=0.238\frac{\omega}{b^{4.2}}\quad,\quad\rho_2=\frac{QL^2}{EI_2\pi^2}\quad,\omega=\left(\frac{a^{10.4}Q}{EI_2}\right)^{0.5}$$

The modified stability functions are:

$$S_1=\left(\omega Lf_4+Za^{5.2}\right)\left(\frac{-LZQb^{5.2}}{\omega PEI_2}\right) \tag{25}$$

$$\overline{SC}=\left(\omega Lf_5+Za^{0.5}b^{4.7}\right)\left(\frac{LZQa^{4.7}b^{0.5}}{\omega PEI_2}\right) \tag{26}$$

$$S_2=\left(\omega Lf_3+Zb^{5.2}\right)\left(\frac{-LZQa^{5.2}}{\omega PEI_2}\right) \tag{27}$$

where

$$P=Z\left[a^{4.7}\left(\frac{f_5}{b^{-0.5}}-\frac{f_3}{a^{-0.5}}\right)-b^{4.7}\left(\frac{f_4}{b^{-0.5}}-\frac{f_6}{a^{-0.5}}\right)\right]-\omega Lf_1f_2 \tag{28}$$

and

$$f_1=J_{-1.119}(\alpha)J_{1.119}(\beta)-J_{1.119}(\alpha)J_{-1.119}(\beta)$$

$$f_2=J_{-0.119}(\alpha)J_{0.119}(\beta)-J_{0.119}(\alpha)J_{-0.119}(\beta)$$

$$f_3 = J_{-0.119}(\alpha)J_{1.119}(\beta) + J_{0.119}(\alpha)J_{-1.119}(\beta)$$

$$f_4 = J_{-0.119}(\beta)J_{1.119}(\alpha) + J_{0.119}(\beta)J_{-1.119}(\alpha)$$

$$f_5 = J_{0.119}(\beta)J_{-1.119}(\beta) + J_{-0.119}(\beta)J_{1.119}(\beta)$$

$$f_6 = J_{-0.119}(\alpha)J_{1.119}(\alpha) + J_{0.119}(\alpha)J_{-1.119}(\alpha)$$

Case 2: Tensile Axial Force Q < 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-10.4}{2} = -4.2\ ,\ n = \frac{1}{2-10.4} = \pm 0.119\ ,\ \beta = \frac{2\omega}{2-10.4} = \pm 0.238\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AI_{0.119}\left(\frac{0.238\omega}{x^{4.2}}\right) + BI_{-0.119}\left(\frac{0.238\omega}{x^{4.2}}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \qquad (29)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{4.7}}AI_{1.119}\left(\frac{0.238\omega}{x^{4.2}}\right) - \frac{\omega}{x^{4.7}}BI_{-1.119}\left(\frac{0.238\omega}{x^{4.2}}\right) - \frac{M_1+M_2}{QL} \qquad (30)$$

The values of the constants A and B are obtained by substituting the boundary conditions into Equation (29):

$$A = \frac{M_1 I_{-0.119}(\alpha)\sqrt{a} + M_2 I_{-0.119}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \qquad (31)$$

$$B = -\frac{M_1 I_{0.119}(\alpha)\sqrt{a} + M_2 I_{0.119}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \qquad (32)$$

where

$$Z = I_{-0.119}(\alpha)I_{0.119}(\beta) - I_{0.119}(\alpha)I_{-0.119}(\beta) \qquad (33)$$

$$\alpha = 0.238\frac{\omega}{a^{4.2}}\ ,\ \beta = 0.238\frac{\omega}{b^{4.2}}\ ,\ \rho_2 = \frac{QL^2}{EI_2\pi^2}\ \text{and}\ \omega = \left(-\frac{a^{10.4}Q}{EI_2}\right)^{0.5}$$

The modified stability functions are: –

$$S_1 = \left(\omega L f_4 + Za^{5.2}\right)\left(\frac{LZQb^{5.2}}{\omega PEI_2}\right) \qquad (34)$$

$$\overline{SC} = \left(\omega L f_5 - Z a^{0.5} b^{4.7}\right)\left(\frac{LZQa^{4.7} b^{0.5}}{\omega PEI_2}\right) \quad (35)$$

$$S_2 = \left(\omega L f_3 + Z b^{5.2}\right)\left(\frac{LZQa^{5.2}}{\omega PEI_2}\right) \quad (36)$$

where

$$P = Z\left[a^{4.7}\left(\frac{f_3}{a^{-0.5}} - \frac{f_5}{b^{-0.5}}\right) + b^{4.7}\left(\frac{f_4}{b^{-0.5}} - \frac{f_6}{a^{-0.5}}\right)\right] - \omega L f_1 f_2 \quad (37)$$

and

$$f_1 = I_{-1.119}(\alpha)I_{1.119}(\beta) - I_{1.119}(\alpha)I_{-1.119}(\beta)$$

$$f_2 = I_{-0.119}(\alpha)I_{0.119}(\beta) - I_{0.119}(\alpha)I_{-0.119}(\beta)$$

$$f_3 = I_{-0.119}(\alpha)I_{1.119}(\beta) - I_{0.119}(\alpha)I_{-1.119}(\beta)$$

$$f_4 = I_{-0.119}(\beta)I_{1.119}(\alpha) - I_{0.119}(\beta)I_{-1.119}(\alpha)$$

$$f_5 = I_{0.119}(\beta)I_{-1.119}(\beta) - I_{-0.119}(\beta)I_{1.119}(\beta)$$

$$f_6 = I_{-0.119}(\alpha)I_{1.119}(\alpha) - I_{0.119}(\alpha)I_{-1.119}(\alpha)$$

**Concave Taper for m = 4 and $\lambda$ = 2.2**

$$\overline{m} = m\overline{\lambda} = 8.8$$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-8.8}{2} = -3.4\ ,\ n = \frac{1}{2-8.8} = \pm 0.147\ ,\ \beta = \frac{2\omega}{2-8.8} = \pm 0.294\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AJ_{0.147}\left(\frac{0.294\omega}{x^{3.4}}\right) + BJ_{-0.147}\left(\frac{0.294\omega}{x^{3.4}}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \quad (38)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{3.9}} AJ_{1.147}\left(\frac{0.294\omega}{x^{3.4}}\right) - \frac{\omega}{x^{3.9}} BJ_{-1.147}\left(\frac{0.294\omega}{x^{3.4}}\right) + \frac{M_1 + M_2}{QL} \quad (39)$$

Thus: -

$$S_1 = \left(\omega Lf_4 + Za^{4.4}\right)\left(\frac{-LZQb^{4.4}}{\omega PEI_2}\right) \tag{40}$$

$$\overline{SC} = \left(\omega Lf_5 + Za^{0.5}\,b^{3.9}\right)\left(\frac{LZQa^{3.9}\,b^{0.5}}{\omega PEI_2}\right) \tag{41}$$

$$S_2 = \left(\omega Lf_3 + Zb^{4.4}\right)\left(\frac{-LZQa^{4.4}}{\omega PEI_2}\right) \tag{42}$$

Case 2: Tensile Axial Force Q < 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-8.8}{2} = -3.4\ ,\ n = \frac{1}{2-8.8} = \pm 0.147\ ,\ \beta = \frac{2\omega}{2-8.8} = \pm 0.294\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AI_{0.147}\left(\frac{0.294\omega}{x^{3.4}}\right) + BI_{-0.147}\left(\frac{0.294\omega}{x^{3.4}}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \tag{43}$$

$$\frac{dy}{dx} = \frac{\omega}{x^{3.9}}AI_{1.147}\left(\frac{0.294\omega}{x^{3.4}}\right) - \frac{\omega}{x^{3.9}}BI_{-1.147}\left(\frac{0.294\omega}{x^{3.4}}\right) - \frac{M_1 + M_2}{QL} \tag{44}$$

Thus: -

$$S_1 = \left(\omega Lf_4 + Za^{4.4}\right)\left(\frac{LZQb^{4.4}}{\omega PEI_2}\right) \tag{45}$$

$$\overline{SC} = \left(\omega Lf_5 - Za^{0.5}\,b^{3.9}\right)\left(\frac{LZQa^{3.9}\,b^{0.5}}{\omega PEI_2}\right) \tag{46}$$

$$S_2 = \left(\omega Lf_3 + Zb^{4.4}\right)\left(\frac{LZQa^{4.4}}{\omega PEI_2}\right) \tag{47}$$

where the equations of $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$, $Z, \alpha, \beta, \omega$, P and the constant of integration are given in **Table (1)**.

**Concave Taper for m = 4 and $\lambda$ = 1.8**

$\overline{m} = m\lambda = 7.2$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-7.2}{2} = -2.6\ ,\ n = \frac{1}{2-7.2} = \pm 0.192\ ,\ \beta = \frac{2\omega}{2-7.2} = \pm 0.385\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AJ_{0.192}\left(\frac{0.385\omega}{x^{2.6}}\right) + BJ_{-0.192}\left(\frac{0.385\omega}{x^{2.6}}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \quad (48)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{3.1}}AJ_{1.192}\left(\frac{0.385\omega}{x^{2.6}}\right) - \frac{\omega}{x^{3.1}}BJ_{-1.192}\left(\frac{0.385\omega}{x^{2.6}}\right) + \frac{M_1+M_2}{QL} \quad (49)$$

The modified stability functions for this case are: –

$$S_1 = \left(\omega Lf_4 + Za^{3.6}\right)\left(\frac{-LZQb^{3.6}}{\omega PEI_2}\right) \quad (50)$$

$$\overline{SC} = \left(\omega Lf_5 + Za^{0.5}b^{3.1}\right)\left(\frac{LZQa^{3.1}b^{0.5}}{\omega PEI_2}\right) \quad (51)$$

$$S_2 = \left(\omega Lf_3 + Zb^{3.6}\right)\left(\frac{-LZQa^{3.6}}{\omega PEI_2}\right) \quad (52)$$

Case 2: Tensile Axial Force Q < 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-7.2}{2} = -2.6\ ,\ n = \frac{1}{2-7.2} = \pm 0.192\ ,\ \beta = \frac{2\omega}{2-7.2} = \pm 0.385\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AI_{0.192}\left(\frac{0.385}{x^{2.6}}\right) + BI_{-0.192}\left(\frac{0.385}{x^{2.6}}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \quad (53)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{3.1}}AI_{1.192}\left(\frac{0.385}{x^{2.6}}\right) - \frac{\omega}{x^{3.1}}BI_{-1.192}\left(\frac{0.385}{x^{2.6}}\right) - \frac{M_1+M_2}{QL} \quad (54)$$

The modified stability functions for this case are: –

$$S_1 = \left(\omega Lf_4 + Za^{3.6}\right)\left(\frac{LZQb^{3.6}}{\omega PEI_2}\right) \quad (55)$$

$$\overline{SC} = \left(\omega Lf_5 - Za^{0.5}b^{3.1}\right)\left(\frac{LZQa^{3.1}b^{0.5}}{\omega PEI_2}\right) \quad (56)$$

$$S_2 = \left(\omega L f_3 + Z b^{3.6}\right)\left(\frac{LZQa^{3.6}}{\omega PEI_2}\right) \quad (57)$$

where the expressions of $f_1, f_2, f_3, f_4, f_5, f_6, Z, \alpha, \beta, \omega$, P and the constant of integration are given in **Table (1).**

**Concave Taper for m = 4 and $\lambda$ = 1.4**

$\overline{m} = m\lambda = 5.6$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-5.6}{2} = -1.8\ ,\ n = \frac{1}{2-5.6} = \pm 0.278\ ,\ \beta = \frac{2\omega}{2-5.6} = \pm 0.556\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AJ_{0.278}\left(\frac{0.556\omega}{x^{1.8}}\right) + BJ_{-0.278}\left(\frac{0.556\omega}{x^{1.8}}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \quad (58)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{2.3}}AJ_{1.278}\left(\frac{0.556\omega}{x^{2.6}}\right) - \frac{\omega}{x^{2.3}}BJ_{-1.278}\left(\frac{0.556\omega}{x^{2.6}}\right) + \frac{M_1 + M_2}{QL} \quad (59)$$

Thus: -

$$S_1 = \left(\omega L f_4 + Z a^{2.8}\right)\left(\frac{-LZQb^{2.8}}{\omega PEI_2}\right) \quad (60)$$

$$\overline{SC} = \left(\omega L f_5 + Z a^{0.5} b^{2.3}\right)\left(\frac{LZQa^{2.3}b^{0.5}}{\omega PEI_2}\right) \quad (61)$$

$$S_2 = \left(\omega L f_3 + Z b^{2.8}\right)\left(\frac{-LZQa^{2.8}}{\omega PEI_2}\right) \quad (62)$$

Case 2: Tensile Axial Force Q < 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-5.6}{2} = -1.8\ ,\ n = \frac{1}{2-5.6} = \pm 0.278\ ,\ \beta = \frac{2\omega}{2-5.6} = \pm 0.556\omega$$

The solution of Equation (12) and its first derivative become:

$$y = x^{0.5}\left[AI_{0.278}\left(\frac{0.556\omega}{x^{1.8}}\right) + BI_{-0.278}\left(\frac{0.556\omega}{x^{1.8}}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \quad (63)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{2.3}} AI_{1.278}\left(\frac{0.556\omega}{x^{2.6}}\right) - \frac{\omega}{x^{2.3}} BI_{-1.278}\left(\frac{0.556\omega}{x^{2.6}}\right) - \frac{M_1 + M_2}{QL} \tag{64}$$

The modified stability functions for this case are: –

$$S_1 = \left(\omega Lf_4 + Za^{2.8}\right)\left(\frac{LZQb^{2.8}}{\omega PEI_2}\right) \tag{65}$$

$$\overline{SC} = \left(\omega Lf_5 - Za^{0.5} b^{2.3}\right)\left(\frac{LZQa^{2.3} b^{0.5}}{\omega PEI_2}\right) \tag{66}$$

$$S_2 = \left(\omega Lf_3 + Zb^{2.8}\right)\left(\frac{LZQa^{2.8}}{\omega PEI_2}\right) \tag{67}$$

where the expressions of $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$, $Z, \alpha, \beta, \omega$, P and the constants of integration are given in **Table (1)**.

## MODIFIED STABILITY FUNCTIONS AT ZERO AXIAL LOAD

The basic differential equation for a tapered beam is ( Al-Damarchi, 1999)

$$EI_2\left(\frac{x}{a}\right)^{\overline{m}} \frac{d^2y}{dx^2} = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \tag{68}$$

This may be written in the form: –

$$EI_2 \frac{d^2y}{dx^2} = \frac{a^{\overline{m}}}{L}\left[(M_1 + M_2)x^{1-\overline{m}} - a(M_1 + uM_2)x^{-\overline{m}}\right]. \tag{69}$$

Integrating with respect to x: –

$$EI_2 \frac{dy}{dx} = \frac{a^{\overline{m}}}{L}\left[(M_1 + M_2)\frac{x^{2-\overline{m}}}{2-\overline{m}} - a(M_1 + uM_2)\frac{x^{1-\overline{m}}}{1-\overline{m}}\right] \tag{70}$$

and again: –

$$EI_2 y = \frac{a^{\overline{m}}}{L}\left[(M_1 + M_2)\frac{x^{3-\overline{m}}}{(2-\overline{m})(3-\overline{m})} - a(M_1 + uM_2)\frac{x^{2-\overline{m}}}{(1-\overline{m})(2-\overline{m})}\right] + Ax + B \tag{71}$$

It may be noted that this solution is used when the value of $\overline{m} \neq 1,2,3$.

There are four unknowns A, B, $M_1$, and $M_2$, which have to be determined from the boundary conditions at the ends. Then, the modified stability functions are:

$$S_1 = \frac{(2-\bar{m})L}{P}\left[\frac{2b}{a^{2\bar{m}-2}}\left(3-4\bar{m}+\bar{m}^2\right)+\frac{3\bar{m}-2-\bar{m}^2}{a^{2\bar{m}-3}}+\frac{2}{a^{\bar{m}}b^{\bar{m}-3}}-\frac{b^2}{a^{2\bar{m}-1}}\left(6-5\bar{m}+\bar{m}^2\right)\right] \quad (72)$$

$$SC = \frac{(\bar{m}-2)L}{P}\left[\frac{b(\bar{m}-3)}{a^{2\bar{m}-2}}+(1-\bar{m})\left(a^{3-2\bar{m}}-\frac{b^{3-\bar{m}}}{a^{\bar{m}}}\right)+\frac{a^{1-\bar{m}}}{b^{\bar{m}-2}}(3-\bar{m})\right] \quad (73)$$

$$S_2 = \frac{(2-\bar{m})L}{P}\left[\frac{2-3\bar{m}+\bar{m}^2}{a^{\bar{m}}b^{\bar{m}-3}}+\frac{8\bar{m}-2\bar{m}^2-6}{a^{\bar{m}-1}b^{\bar{m}-2}}-\frac{2}{b^{2\bar{m}-3}}+\frac{a^{1-\bar{m}}}{b^{\bar{m}-2}}(3-\bar{m})\right] \quad (74)$$

where

$$P = \frac{2b^{2-\bar{m}}}{a^{\bar{m}-2}}\left(3-4\bar{m}+\bar{m}^2\right)+\left(\frac{b^{1-\bar{m}}}{a^{\bar{m}-3}}+\frac{b^{3-\bar{m}}}{a^{\bar{m}-1}}\right)\left(4-4\bar{m}+\bar{m}^2\right)+b^{4-2\bar{m}}+a^{4-2\bar{m}} \quad (75)$$

**EXAMPLE**

By comparing between the elastic critical loads of the frame shown in **Fig. (2)** for the two cases of stanchion shapes, the more efficient shape can be found. The first case shows a linear taper member while the second depicts a concave taper with the other properties being the same.

For the non-sway mode and joint B rotating by $\theta_B$:

$$K_{BC} = \frac{EI}{L} \quad , K_{AB} = \frac{EI_A}{L}$$

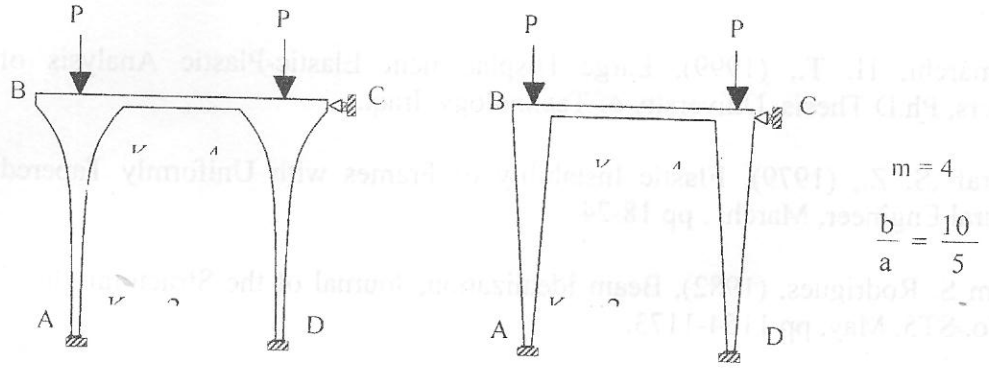



Fig. (2)

$$\begin{aligned}\sum M_B &= M_{BC} + M_{BA}\\ &= K_{BC}\left[S\theta_B + SC\theta_C\right] + K_{BA}\left[S_1\theta_B + \overline{SC}\theta_A\right]\\ &= 4\left[4+2\right]\theta_B + 2S_1\theta_B\\ &= \left(24+2S_1\right)\theta_B\end{aligned}$$

where

S = 4 and SC = 2 for prismatic member and zero axial load ( Livesley, 1956).

$\theta_B = \theta_C$ and $\theta_A, \theta_D = 0$

The critical load makes the stiffness vanishes.

$24 + 2S_1 = 0$

$S_1 = -12$

By trials, the different values of $\rho$ are substituted in the stability equation for $S_1$ until $S_1 = -12$ for each $\lambda$ factor.

| $\overline{m}$ | 4 | 5.6 | 7.2 | 8.8 | 10.4 |
|---|---|---|---|---|---|
| $\lambda$ | 1.0 | 1.4 | 1.8 | 2.2 | 2.6 |
| $\rho$ | 9.783 | 14.228 | 20.371 | 28.467 | 38.450 |

By comparing the critical $\rho$ from the concave taper strut with that of the linear taper, it is seen that the concave shape is more efficient in carrying loads over the linear taper member.

## CONCLUSIONS

Modified stability functions are derived for non-linear concave taper members having a shape factor m = 4 for a square or circular solid cross-sectional area. The non-linearity concave factor $\lambda$ = 1.4, 1.8, 2.2 and 2.6 are subjected to a constant compressive, tensile or zero axial load obtaining closed form solutions.

These expressions for the modified stability functions $S_1, \overline{SC}$ and $S_2$ are exact for any depth ratio and any non-dimensional axial load parameter $\rho$. The concave shape of the beam-column may enhance the strength of the strut.

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## SYMBOLS

| | |
|---|---|
| E | : Young's modulus |
| $f_1, f_2, \ldots f_6$ | : Parametric functions |
| $I_1$ | : Moment of inertia at end 1 |
| $I_2$ | : Moment of inertia at end 2 |
| K | : Stiffness of the strut |
| L | : Member length |
| m | : Shape factor |
| $\overline{m}$ | : Modified Shape factor |
| M1, M2 | : Bending moments at ends 1 and 2 of the member |
| Q | : Axial load |
| $Q_E$ | : Euler load = $EI_2L^2/\pi^2$ (for end 2) |
| $S_1$ | : Modified stability function at end 1 |
| $S_2$ | : Modified stability function at end 2 |
| SC | : Carrying factor of the modified stability function |
| U | ; Modified taper ratio |
| Y | : Deflection |
| $\lambda$ | : Non-linearity factor |
| $\theta_1, \theta_2$ | : Rotations at ends 1 and 2 of member |
| $\rho$ | : Axial load parameter = $Q/Q_E$ |

Table (1) Formulae of derivation symbols in stability function

| | Symbol | $\overline{m} = 8.8 \quad \lambda = 2.2$ | $\overline{m} = 7.2 \quad \lambda = 1.8$ | $\overline{m} = 5.6 \quad \lambda = 1.4$ |
|---|---|---|---|---|
| Compressive Axial Force, $Q > 0$ | Z | $J_{0.147}(\alpha)J_{-0.147}(\beta) - J_{-0.147}(\alpha)J_{0.147}(\beta)$ | $J_{-0.192}(\alpha)J_{0.192}(\beta) - J_{0.192}(\alpha)J_{-0.192}(\beta)$ | $J_{0.278}(\alpha)J_{-0.278}(\beta) - J_{-0.278}(\alpha)J_{0.278}(\beta)$ |
| | ω | $(a^{8.8} Q/EI_2)^{0.5}$ | $(a^{7.2} Q/EI_2)^{0.5}$ | $(a^{5.6} Q/EI_2)^{0.5}$ |
| | α | $0.294\omega a^{-3.4}$ | $0.385\omega a^{-2.6}$ | $0.556\omega a^{-1.8}$ |
| | β | $0.294\omega b^{-3.4}$ | $0.385\omega b^{-2.6}$ | $0.556\omega b^{-1.8}$ |
| | A | $\frac{M_1 J_{-0.147}(\alpha)\sqrt{a} + M_2 J_{-0.147}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $\frac{M_1 J_{-0.192}(\alpha)\sqrt{a} + M_2 J_{-0.192}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $\frac{M_1 J_{-0.278}(\alpha)\sqrt{a} + M_2 J_{-0.278}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ |
| | B | $-\frac{M_1 J_{0.147}(\alpha)\sqrt{a} + M_2 J_{0.147}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $-\frac{M_1 J_{0.192}(\alpha)\sqrt{a} + M_2 J_{0.192}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $-\frac{M_1 J_{0.278}(\alpha)\sqrt{a} + M_2 J_{0.278}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ |
| | $f_1$ | $J_{-1.147}(\alpha)J_{1.147}(\beta) - J_{1.147}(\alpha)J_{-1.147}(\beta)$ | $J_{-1.192}(\alpha)J_{1.192}(\beta) - J_{1.192}(\alpha)J_{-1.192}(\beta)$ | $J_{-1.278}(\alpha)J_{1.278}(\beta) - J_{1.278}(\alpha)J_{-1.278}(\beta)$ |
| | $f_2$ | $J_{-0.147}(\alpha)J_{0.147}(\beta) - J_{0.147}(\alpha)J_{-0.147}(\beta)$ | $J_{-0.192}(\alpha)J_{0.192}(\beta) - J_{0.192}(\alpha)J_{-0.192}(\beta)$ | $J_{-0.278}(\alpha)J_{0.278}(\beta) - J_{0.278}(\alpha)J_{-0.278}(\beta)$ |
| | $f_3$ | $J_{-0.147}(\alpha)J_{1.147}(\beta) + J_{0.147}(\alpha)J_{-1.147}(\beta)$ | $J_{-0.192}(\alpha)J_{1.192}(\beta) + J_{0.192}(\alpha)J_{-1.192}(\beta)$ | $J_{-0.278}(\alpha)J_{1.278}(\beta) + J_{0.278}(\alpha)J_{-1.278}(\beta)$ |
| | $f_4$ | $J_{-0.147}(\beta)J_{1.147}(\alpha) + J_{0.147}(\beta)J_{-1.147}(\alpha)$ | $J_{-0.192}(\beta)J_{1.192}(\alpha) + J_{0.192}(\beta)J_{-1.192}(\alpha)$ | $J_{-0.278}(\beta)J_{1.278}(\alpha) + J_{0.278}(\beta)J_{-1.278}(\alpha)$ |
| | $f_5$ | $J_{0.147}(\beta)J_{-1.147}(\beta) + J_{-0.147}(\beta)J_{1.147}(\beta)$ | $J_{0.192}(\beta)J_{-1.192}(\beta) + J_{-0.192}(\beta)J_{1.192}(\beta)$ | $J_{0.278}(\beta)J_{-1.278}(\beta) + J_{-0.278}(\beta)J_{1.278}(\beta)$ |
| | $f_6$ | $J_{-0.147}(\alpha)J_{1.147}(\alpha) + J_{0.147}(\alpha)J_{-1.147}(\alpha)$ | $J_{-0.192}(\alpha)J_{1.192}(\alpha) + J_{0.192}(\alpha)J_{-1.192}(\alpha)$ | $J_{-0.278}(\alpha)J_{1.278}(\alpha) + J_{0.278}(\alpha)J_{-1.278}(\alpha)$ |
| | P1 | $(f_5 b^{0.5} - f_3 a^{0.5})$ | $(f_5 b^{0.5} - f_3 a^{0.5})$ | $(f_5 b^{0.5} - f_3 a^{0.5})$ |
| | P2 | $(f_4 b^{0.5} - f_6 a^{0.5})$ | $(f_4 b^{0.5} - f_6 a^{0.5})$ | $(f_4 b^{0.5} - f_6 a^{0.5})$ |
| | P | $Z[a^{3.9} P_1 - b^{3.9} P_2] - \omega L f_1 f_2$ | $Z[a^{3.1} P_1 - b^{3.1} P_2] - \omega L f_1 f_2$ | $Z[a^{2.3} P_1 - b^{2.3} P_2] - \omega L f_1 f_2$ |
| Tensile Axial Force, $Q < 0$ | Z | $I_{0.147}(\alpha)I_{-0.147}(\beta) - I_{-0.147}(\alpha)I_{0.147}(\beta)$ | $I_{-0.192}(\alpha)I_{0.192}(\beta) - I_{0.192}(\alpha)I_{-0.192}(\beta)$ | $I_{-0.278}(\alpha)I_{0.278}(\beta) - I_{0.278}(\alpha)I_{-0.278}(\beta)$ |
| | ω | $(-a^{8.8} Q/EI_2)^{0.5}$ | $(-a^{7.2} Q/EI_2)^{0.5}$ | $(-a^{5.6} Q/EI_2)^{0.5}$ |
| | α | $0.294\omega a^{-3.4}$ | $0.385\omega a^{-2.6}$ | $0.556\omega a^{-1.8}$ |
| | β | $0.294\omega b^{-3.4}$ | $0.385\omega b^{-2.6}$ | $0.556\omega b^{-1.8}$ |
| | A | $\frac{M_1 I_{-0.147}(\alpha)\sqrt{a} + M_2 I_{-0.147}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $\frac{M_1 I_{-0.192}(\alpha)\sqrt{a} + M_2 I_{-0.192}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $\frac{M_1 I_{-0.278}(\alpha)\sqrt{a} + M_2 I_{-0.278}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ |
| | B | $-\frac{M_1 I_{0.147}(\alpha)\sqrt{a} + M_2 I_{0.147}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $-\frac{M_1 I_{0.192}(\alpha)\sqrt{a} + M_2 I_{0.192}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ | $-\frac{M_1 I_{0.278}(\alpha)\sqrt{a} + M_2 I_{0.278}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b} ZQ}$ |
| | $f_1$ | $I_{-1.147}(\alpha)I_{1.147}(\beta) - I_{1.147}(\alpha)I_{-1.147}(\beta)$ | $I_{-1.192}(\alpha)I_{1.192}(\beta) - I_{1.192}(\alpha)I_{-1.192}(\beta)$ | $I_{-1.278}(\alpha)I_{1.278}(\beta) - I_{1.278}(\alpha)I_{-1.278}(\beta)$ |
| | $f_2$ | $I_{-0.147}(\alpha)I_{0.147}(\beta) - I_{0.147}(\alpha)I_{-0.147}(\beta)$ | $I_{-0.192}(\alpha)I_{0.192}(\beta) - I_{0.192}(\alpha)I_{-0.192}(\beta)$ | $I_{-0.278}(\alpha)I_{0.278}(\beta) - I_{0.278}(\alpha)I_{-0.278}(\beta)$ |
| | $f_3$ | $I_{-0.147}(\alpha)I_{1.147}(\beta) + I_{0.147}(\alpha)I_{-1.147}(\beta)$ | $I_{-0.192}(\alpha)I_{1.192}(\beta) + I_{0.192}(\alpha)I_{-1.192}(\beta)$ | $I_{-0.278}(\alpha)I_{1.278}(\beta) + I_{0.278}(\alpha)I_{-1.278}(\beta)$ |
| | $f_4$ | $I_{-0.147}(\beta)I_{1.147}(\alpha) + I_{0.147}(\beta)I_{-1.147}(\alpha)$ | $I_{-0.192}(\beta)I_{1.192}(\alpha) + I_{0.192}(\beta)I_{-1.192}(\alpha)$ | $I_{-0.278}(\beta)I_{1.278}(\alpha) + I_{0.278}(\beta)I_{-1.278}(\alpha)$ |
| | $f_5$ | $I_{0.147}(\beta)I_{-1.147}(\beta) + I_{-0.147}(\beta)I_{1.147}(\beta)$ | $I_{0.192}(\beta)I_{-1.192}(\beta) + I_{-0.192}(\beta)I_{1.192}(\beta)$ | $I_{0.278}(\beta)I_{-1.278}(\beta) + I_{-0.278}(\beta)I_{1.278}(\beta)$ |
| | $f_6$ | $I_{-0.147}(\alpha)I_{1.147}(\alpha) + I_{0.147}(\alpha)I_{-1.147}(\alpha)$ | $I_{-0.192}(\alpha)I_{1.192}(\alpha) + I_{0.192}(\alpha)I_{-1.192}(\alpha)$ | $I_{-0.278}(\alpha)I_{1.278}(\alpha) + I_{0.278}(\alpha)I_{-1.278}(\alpha)$ |
| | $P_1$ | $(f_5 b^{0.5} + f_3 a^{0.5})$ | $(f_5 b^{0.5} + f_3 a^{0.5})$ | $(f_5 b^{0.5} + f_3 a^{0.5})$ |
| | $P_2$ | $\rho = Q/Q_E$ | $(f_4 b^{0.5} - f_6 a^{0.5})$ | $(f_4 b^{0.5} - f_6 a^{0.5})$ |
| | P | $Z[a^{3.9} P_1 + b^{3.9} P_2] - \omega L f_1 f_2$ | $Z[a^{3.1} P_1 + b^{3.1} P_2] - \omega L f_1 f_2$ | $Z[a^{2.3} P_1 + b^{2.3} P_2] - \omega L f_1 f_2$ |